

Laplacian ISI Eigenvalue Based Anomaly Detection in IoT Networks

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Abstract Anomaly detection in Internet of Things networks is a critical challenge due to dynamic topology, heterogeneous devices, and complex communication patterns. This paper presents a novel spectral framework based on the Laplacian Inverse Sum In-degree matrix for detecting anomalies in graph-structured data. The proposed approach models IoT systems as graphs and extracts Laplacian ISI eigenvalues as discriminative features that capture both structural connectivity and degree-based relationships. Unlike traditional Laplacian methods, the ISI formulation provides enhanced sensitivity to structural perturbations. The extracted spectral features are integrated with machine learning models for anomaly classification. Experimental analysis on simulated network data demonstrates that deviations in the Laplacian ISI spectrum effectively identify abnormal behavior such as node failures and malicious activities. The proposed framework bridges spectral graph theory and practical anomaly detection, making it suitable for cybersecurity and IoT applications.

1 Introduction

The rapid growth of Internet of Things (IoT) systems has led to increasingly complex and interconnected network structures. These systems are highly vulnerable to anomalies such as intrusions, node failures, and abnormal communication patterns. Detecting such anomalies

is essential for ensuring network reliability and security. Graph-based representations have emerged as a powerful tool for modeling IoT systems, where nodes represent devices and edges represent communication links. Spectral graph theory, particularly Laplacian-based methods, has been widely used to analyze structural properties of such networks. However, traditional Laplacian matrices do not fully capture degree-sensitive interactions among nodes. To address this limitation, this paper utilizes the Laplacian ISI matrix, which incorporates inverse sum in-degree information into the graph structure. The eigenvalues of this matrix provide a rich representation of network behavior and can be used to detect anomalies through spectral deviations.

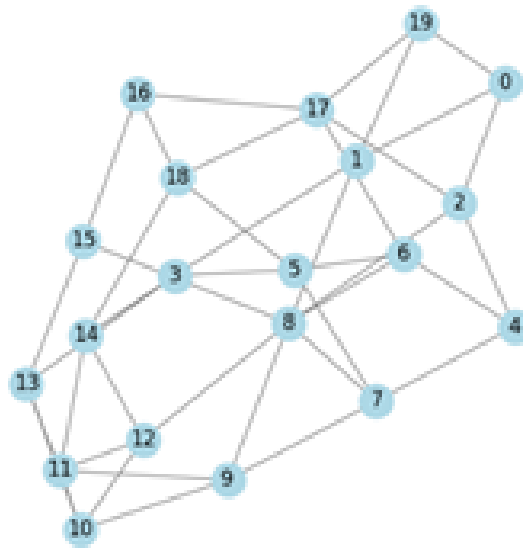


Figure 1: Normal IoT Network Topology Based on Small Communication Model

The figure 1 illustrates the structural representation of a normal IoT network modeled using a small-world graph topology, where nodes represent IoT devices such as sensors, gateways, and actuators, and edges represent legitimate communication links among them. The network exhibits high clustering with short average path length, which is a key characteristic of real-world IoT deployments enabling efficient local communication with occasional long-range interactions. In the normal state, the connectivity pattern remains stable and well-distributed, without any abnormal or irregular link formations. This structural regularity ensures balanced information flow and maintains the integrity of the underlying network topology. From a spectral graph perspective, such a configuration produces a smooth Laplacian eigenvalue distribution, reflecting stable diffusion dynamics and consistent node degree relationships across the network.

The figure 2 represents an IoT network under attack, where the original small-world topology is disrupted through the introduction of malicious edges simulating unauthorized communication or compromised device behavior. These additional connections create structural anomalies by altering the local neighborhood structure and disturbing the inherent clustering properties of the network. Such perturbations increase irregularity in node de-

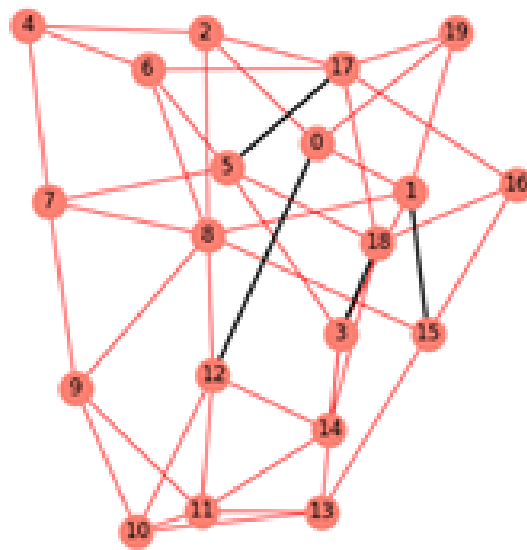


Figure 2: IoT Network Under Attack with Malicious Edge Injection and Structural Perturbation

gree distribution and introduce unexpected connectivity paths between distant or unrelated nodes. From a cybersecurity perspective, this scenario models common IoT threats such as device spoofing, lateral movement, or injection of rogue communication links. In spectral terms, these structural modifications lead to measurable distortions in the Laplacian eigenvalue spectrum, indicating a deviation from the normal network state and providing a strong signal for anomaly detection.

The main contributions of this work are as follows:

- Extending Laplacian ISI eigenvalues to anomaly detection in IoT networks
- Developing a spectral feature extraction framework
- Integrating machine learning for anomaly classification
- Demonstrating effectiveness through experimental evaluation

2 Related Work

Anomaly detection in graph-structured data has gained significant attention due to its applications in cybersecurity, Internet of Things (IoT), and social networks. Traditional anomaly detection techniques, including statistical and distance-based approaches, often fail to capture complex structural dependencies in graph data. As a result, graph-based anomaly detection methods have emerged as a powerful alternative, leveraging connectivity patterns and structural relationships to identify abnormal behavior [22].

Spectral graph theory provides an effective framework for analyzing graph structures through eigenvalues and eigenvectors of Laplacian matrices. Spectral methods have been widely used for detecting anomalies by identifying deviations in graph structure and connectivity patterns. These approaches exploit the fact that anomalies often introduce per-

turbations in the spectral domain, leading to measurable changes in eigenvalue distributions [23]. However, conventional Laplacian-based methods are limited in their ability to incorporate degree-sensitive information, which may reduce their effectiveness in detecting subtle anomalies.

Recent advancements have focused on integrating spectral methods with machine learning techniques. In particular, graph-based anomaly detection frameworks often follow a two-stage process involving feature extraction and anomaly classification [14]. Spectral features derived from graph representations have been shown to improve detection accuracy by capturing both global and local structural properties.

With the rapid development of deep learning, Graph Neural Networks (GNNs) have become a dominant approach for anomaly detection in graph data. GNN-based models learn node embeddings by aggregating neighborhood information and have demonstrated strong performance in detecting anomalous nodes and edges [9]. Several studies have proposed advanced models that combine spatial and spectral information to improve anomaly detection performance, particularly in heterogeneous and dynamic graphs [25].

In the context of IoT and network security, anomaly detection has become increasingly important due to the presence of large-scale, dynamic, and heterogeneous networks. Recent works have proposed graph-based intrusion detection systems that transform network traffic into graph structures and apply machine learning or deep learning techniques for identifying malicious behavior [1]. These approaches have shown promising results in detecting cyber-attacks and abnormal communication patterns.

Furthermore, recent research has explored spectral-spatial learning frameworks that integrate graph structure and feature information to enhance anomaly detection. These methods leverage both local and global graph properties to improve robustness and detection accuracy [23]. Additionally, autoencoder-based and transformer-based models have been proposed for anomaly detection in complex systems, demonstrating the effectiveness of representation learning techniques [5].

Despite these advancements, most existing approaches rely on classical Laplacian matrices and do not consider alternative formulations that incorporate degree-sensitive information. The Laplacian Inverse Sum In-degree (ISI) matrix provides a more expressive representation of graph structure by integrating degree-based interactions. However, its application in anomaly detection remains largely unexplored.

Therefore, this study aims to address this gap by leveraging Laplacian ISI eigenvalues as discriminative spectral features for anomaly detection. The proposed approach provides a lightweight and interpretable alternative to deep learning-based methods while maintaining high sensitivity to structural variations in complex networks.

3 Preliminaries and Mathematical Model

A graph, denoted by the notation $G = (V, E)$, is composed of a collection of vertices, also known as $V(G) = \{v_1, v_2, \dots, v_n\}$, and a collection of edges, denoted by the notation $E(G)$. In this context, we solely consider basic, undirected graphs unless otherwise specified. The number of vertices in $V(G)$ is referred to as the order n , whereas the quantity of edges in $E(G)$ is termed the size m of G . When we denote $u \sim v$, we signify that vertex u is next

to vertex v , and we represent the edge connecting u and v as e . The neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of all vertices that are adjacent to v . The degree of a vertex v_i , represented by $d(v_i)$ (or d_i), is the number of vertices in the neighborhood $N(v_i)$. A graph G is classified as r -regular if every vertex in G has a degree of r . In a connected graph (G), the distance $d(u, v)$ between two distinct vertices (u, v) is defined as the length of the shortest path that joins them. The largest distance that may exist between any two vertices in a graph is referred to as the diameter of the graph G . We designate the complete graph on n vertices as K_n , The complete bipartite graph with partition sizes a and b is denoted as $K_{a,b}$, whereas the star graph is represented as $K_{1,n-1}$. The union of two graphs G_1 and G_2 , denoted as $G_1 \cup G_2$, consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The vertex-disjoint union of k copies of G is $k \cdot G$. Standard graph theory notation is followed, with additional notations provided in [6].

The Inverse Sum Indegree Index, often known as the ISI index [24], is a topological index that is defined by the following rules:

$$ISI(G) = \sum_{v_i v_j \in E(G)} \frac{d_{v_i} d_{v_j}}{d_{v_i} + d_{v_j}}$$

This index is well examined and has several applications in quantitative structure-activity relationships (QSAR) and quantitative structure-property relationships (QSPR) [15, 17, 13]. According to Zangi et al. [27], the inverse sum indeg matrix (ISI matrix) of a graph G is defined as a $n \times n$ square matrix. This matrix is a representation of the inverse sum of degree matrices.

$$A_{ISI}(G) = (s_{ij})_{n \times n} = \begin{cases} \frac{d_{v_i} d_{v_j}}{d_{v_i} + d_{v_j}} & \text{if there is an edge between } v_i \text{ and } v_j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A_{ISI}(G)$ for a graph G are termed the *ISI* eigenvalues of G . Let $s_1 \geq s_2 \geq \dots \geq s_n$ denote the ISI-eigenvalues of G , with s_1 referred to as the ISI-spectral radius of G . In 2015, Li [16] introduced the concept of examining matrices derived from topological or chemical indices.

Sedlar et al. [21] conduct an investigation on the characteristics of the ISI index and determine the values of the ISI index that are considered to be extreme for specific types of graphs. Extremal constraints on the ISI index were provided by Pattabiraman [18].

In 2018, Das et al. [7] summarized many forms of graph energies introduced by numerous authors. Das et al. [7] establish several lower and upper constraints for the energies of graphs. To obtain results regarding various energy types of graphs, one may consult [4, 8, 10, 19, 20, 28]. Zangi et al. [27] proposed the notion of ISI energy for graphs. Hafeez and Farooq [11] subsequently constructed the ISI energy formulas for many prominent graphs and examined both lower and upper bounds for ISI energy. Hatefi et al. [12] established that a star graph possesses the lowest ISI energy. Ye and Li [26] investigated ISI-noncospectral equienergetic graphs of order beyond 8.

For a graph G , the Laplacian matrix is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G , and $A(G)$ is the adjacency matrix of G . The eigenvalues of the Laplacian matrix are denoted as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. The largest

eigenvalue of the Laplacian matrix of graph G is termed its Laplacian spectral radius. Since the Laplacian matrix of a graph provides greater insight into graph structures than its adjacency matrix, it is reasonable to extend the ISI-adjacency matrix of G to the Laplacian ISI-matrix. We defined the Laplacian ISI-matrix for a simple graph in this paper. As part of our research, we investigate a few features of the laplacian ISI matrix and discover the laplacian ISI eigenvalues of a few families of graphs. We characterize all graphs with two (respectively, three) distinct Laplacian ISI-eigenvalues.

4 Theoretical Results

4.1 Laplacian ISI-matrix

For each vertex v_i in the vertex set of graph G , let $\overline{d_{v_i}}$ represent the summation of $\frac{d_{v_i}^2}{d_{v_i} + d_{v_j}}$ for every vertex v_j in the neighborhood $N(v_i)$. For a graph G of order n , the Laplacian ISI-matrix is defined as $\mathcal{L}(G) = \overline{D}(G) - A_{ISI}(G)$, where $\overline{D}(G) = [\overline{d_{ii}}]_{n \times n}$ is the ISI diagonal matrix with diagonal entries $\overline{d_{ii}} = \overline{d_{v_i}} = \sum_{v_j \in N(v_i)} \frac{d_{v_i}^2}{d_{v_i} + d_{v_j}}$, which denotes the ISI-degree of vertex v_i . Henceforth, we shall denote d_i in place of $\overline{d_{v_i}}$.

Let G be an n -vertex graph. Then the eigenvalues of $\mathcal{L}(G)$ are represented as $\eta_1 \geq \eta_2 \geq \dots \geq \eta_n$, referred to as the Laplacian ISI-eigenvalues of G . The greatest eigenvalue of the Laplacian ISI-matrix of G constitutes the Laplacian ISI-spectral radius of G .

Let G be an n -vertex graph with m edges, and let D be a directed graph. We define the vertex-arc incidence matrix $S(G) = [s_{ve}]_{n \times m}$, where the element s_{ve} for vertex v and arc e is defined as:

$$s_{ve} = \begin{cases} \frac{d_{v_i}}{\sqrt{d_{v_i} + d_{v_j}}} & \text{if } v \text{ is the terminal vertex of } e, \\ \frac{-d_{v_j}}{\sqrt{d_{v_i} + d_{v_j}}} & \text{if } v \text{ is the initial vertex of } e, \\ 0 & \text{if } v \text{ and } e \text{ are not incident.} \end{cases} \quad (4.1)$$

Theorem 4.1. *Let G be a graph of order n and D be its directed graph. Then for any orientation of the graph, we have $\mathcal{L}(G) = S(G) \cdot S(G)'$, where $S(G)$ is defined in (4.1) and $S(G)' = [s_{ve}]^t$, the transpose of $S(G)$.*

Proof. Let $[s_{ve}] \cdot [s_{ve}]^t = (z_{kl})_{n \times n}$. We will analyze the correlation between the components of $\mathcal{L}(G)$ and $S(G) \cdot S(G)'$. It is evident that $z_{kl} = \sum_{e \in E} s_{ke} \cdot s_{le}$.

If $k = l$, then

$$z_{kl} = \sum_{e \in E} s_{ke} \cdot s_{le} = \sum_{v_k v_l = e \in E} \frac{d_k^2}{d_k + d_l}.$$

If $k \neq l$, then $z_{kl} = s_{ke} \cdot s_{le}$, where v_k and v_l are adjacent and $s_{ke} = \frac{d_k}{\sqrt{d_k + d_l}}$ and $s_{le} = -\frac{d_l}{\sqrt{d_k + d_l}}$. This results in:

$$z_{kl} = \frac{-d_k d_l}{d_k + d_l}.$$

In conjunction with the preceding reasoning, we have $\mathcal{L}(G) = S(G) \cdot S(G)'$. □

Lemma 4.2. For any graph G , demonstrate that $\mathcal{L}(G)$ is a positive semi-definite matrix.

Proof. Let $x = (x_u)$ ($u \in V(G)$) be a column vector. We have

$$x^t L(G) x = x^t [s_{ke}] [s_{ke}] x = \sum_{e=v_k v_l} \frac{(d_k x_k - d_l x_l)^2}{d_k + d_l} \geq 0.$$

This completes the proof. $\square$

Theorem 4.3. Let G be an n -vertex connected graph. Then the Laplacian matrix $\mathcal{L}(G)$ is singular, and the singular value is associated with the eigenvector $x = \left(1, \frac{d_1}{d_2}, \frac{d_1}{d_3}, \dots, \frac{d_1}{d_n}\right)$.

Proof. Let $x = (x_1, x_2, \dots, x_n)^t$ be a non-zero vector. Then for any non-zero vector x , we have $\mathcal{L}x = 0$ if and only if $S'x = 0$. Subsequently, for any edge $e = v_k v_l$, we have

$$(S'(G)x)_e = \frac{d_k}{\sqrt{d_k + d_l}} x_k - \frac{d_l}{\sqrt{d_k + d_l}} x_l, \text{ for any edge } e = v_k v_l.$$

We observe that $S'x = 0$ necessitates that $\frac{d_k}{\sqrt{d_k + d_l}} x_k - \frac{d_l}{\sqrt{d_k + d_l}} x_l = 0$. This results in $d_k x_k - d_l x_l = 0$, which consequently indicates that $x_l = \frac{d_k}{d_l} x_k$. We possess

$$x_1 = \frac{d_2}{d_1} x_2 = \frac{d_2}{d_1} \cdot \frac{d_3}{d_2} x_3 = \frac{d_2}{d_1} \cdot \frac{d_3}{d_2} \cdot \frac{d_4}{d_3} x_4 = \dots = \frac{d_2}{d_1} \cdot \frac{d_3}{d_2} \cdot \frac{d_4}{d_3} \cdot \dots \cdot \frac{d_n}{d_{n-1}} x_n.$$

This results in $x_i = \frac{d_1}{d_i} x_1$ for $i = 2, \dots, n$. This concludes the proof. $\square$

This lemma provides the rank of the incidence matrix; the proof is analogous to the proof of Lemma 2.2[2].

Lemma 4.4. Let G be a connected n -vertex graph, where $n \geq 3$. Then rank of the incidence matrix is $n - 1$.

Lemma 4.5. Let G be an n -vertex graph, with $n \geq 3$. Then $\mathcal{L}(G)$ possesses k distinct eigenvalues, where $2 \leq k \leq n$, if and only if there exist $k - 1$ distinct nonzero numbers $s_1, s_2, \dots, s_{k-1}$ such that:

$$\prod_{j=1}^{k-1} (\mathcal{L}(G) - s_j I) = \frac{(-1)^{k-1} \prod_{j=1}^{k-1} s_j}{\left(1 + \frac{d_1^2}{d_2^2} + \frac{d_1^2}{d_3^2} + \dots + \frac{d_1^2}{d_n^2}\right)} K, \quad (4.2)$$

where K represents a square matrix of order n , with its i -th column expressed as $\frac{d_1}{d_i} \cdot z$, where z is a vector as specified in Theorem 4.3 and I is the identity matrix of order n .

Proof. We commence by demonstrating the direct component. Assuming that (4.2) is valid, we subsequently multiply both sides of (4.2) by $\mathcal{L}(G)$, resulting in:

$$\mathcal{L}(G) \prod_{j=1}^{k-1} (\mathcal{L}(G) - s_j I) = \frac{(-1)^{k-1} \prod_{j=1}^{k-1} s_j}{\left(1 + \frac{d_1^2}{d_2^2} + \frac{d_1^2}{d_3^2} + \dots + \frac{d_1^2}{d_n^2}\right)} \mathcal{L}(G) K.$$

It is clear that $\mathcal{L}(G)K = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector containing n elements, so the equation reduces to:

$$\mathcal{L}(G) (\mathcal{L}(G) - s_1 I) (\mathcal{L}(G) - s_2 I) \dots (\mathcal{L}(G) - s_{k-1} I) = \mathbf{0}.$$

By employing the notion of the minimal-polynomial $p(x)$ of a matrix, we ascertain that the minimal-polynomial of the matrix $\mathcal{L}(G)$ is:

$$p(x) = x(x - s_1)(x - s_2) \dots (x - s_{k-1})$$

Thus, $\mathcal{L}(G)$ possesses k distinct eigenvalues: $0, s_1, s_2, \dots, s_{k-1}$.

Conversely, if $0, s_1, s_2, \dots, s_{k-1}$ are the k distinct eigenvalues of $\mathcal{L}(G)$, with each $s_j \neq 0$, then the minimum polynomial is explicitly defined as $p(x) = x(x - 1)(x - 2) \dots (x - l)$, resulting in the following implication:

$$\mathcal{L}(G) \prod_{j=1}^{k-1} (\mathcal{L}(G) - s_j I) = \mathbf{0}.$$

If we assume that the graph G is connected, then any eigenvector of $\mathcal{L}(G)$ that corresponds to the zero eigenvalue is a scalar multiple of the vector $z = \left(1, \frac{d_1}{d_2}, \frac{d_1}{d_3}, \dots, \frac{d_1}{d_n}\right)$. Thus, the i -th column vector of the matrix $\prod_{j=1}^{k-1} (\mathcal{L}(G) - s_j I)$ can be represented as $c_i z$ for a certain scalar c_i (where $i = 1, 2, \dots, n$). Consequently, we derive:

$$\prod_{j=1}^{k-1} (\mathcal{L}(G) - s_j I) = z^t (c_1, c_2, \dots, c_n).$$

The following is the result of multiplying both sides of the equality given above by z :

$$(-1)^{k-1} \prod_{j=1}^{k-1} s_j z = \left(1 + \frac{d_1^2}{d_2^2} + \frac{d_1^2}{d_3^2} + \dots + \frac{d_1^2}{d_n^2}\right) (c_1, c_2, \dots, c_n).$$

For all integers i satisfying $1 \leq i \leq n$, it is evident that

$$c_i = \frac{(-1)^{k-1} d_1}{d_i \left(1 + \frac{d_1^2}{d_2^2} + \frac{d_1^2}{d_3^2} + \dots + \frac{d_1^2}{d_n^2}\right)} \prod_{j=1}^{k-1} s_j.$$

This completes the proof. □

Theorem 4.6. *Let G be an n -vertex graph. Then the graph G possesses precisely one different Laplacian ISI-eigenvalue if and only if $G = nK_1$.*

Proof. Observe that $\mathcal{L}(G)$ is positive semi-definite. The trace of $\mathcal{L}(G)$ equals zero if and only if $\mathcal{L}(G)$ is zero, which occurs alone when the degree of each vertex is zero. Consequently, the outcome ensues. □

Theorem 4.7. *Let G be a connected n -vertex graph with $n \geq 2$. Then the graph G possesses precisely two distinct eigenvalues if and only if G is equal to K_n .*

Proof. According to Lemma 4.5, graph G exhibits exactly two distinct eigenvalues if and only if there exists a non-zero scalar s such that

$$\mathcal{L}(G) - sI = \frac{-s}{\left(1 + \frac{d_1^2}{d_2^2} + \frac{d_1^2}{d_3^2} + \dots + \frac{d_1^2}{d_n^2}\right)} K$$

If G is the same as K_n , it is evident that $\left(1 + \frac{d_1^2}{d_2^2} + \frac{d_1^2}{d_3^2} + \dots + \frac{d_1^2}{d_n^2}\right) = n$ and $K = J$, where J denotes the all-ones matrix of order $n \times n$. The aforementioned equation transforms into:

$$\mathcal{L}(G) - sI = \frac{-s}{n} J.$$

This implies that $s = \frac{n(n-1)}{2}$. □

Theorem 4.8. *Let G be an n -vertex connected graph with $n \geq 3$. Let G possess s connected components, each containing n_i vertices, where $n_i \geq 3$ for $i = 1, 2, \dots, s$. Then, given $\mathcal{L}(G)$, the number of components is equal to the multiplicity of the eigenvalue 0.*

Proof. It is simple to obtain that

$$\text{rank}(\mathcal{L}(G)) = \text{rank}(R(G)R(G)^t) = \text{rank}(R(G)).$$

Lemma 4.4 states that the rank of $\mathcal{L}(G)$ and the rank of $R(G)$ are both equal to $n - 1$ if G is connected. Given that $\mathcal{L}(G)$ is a real symmetric matrix, the eigenvalue 0 has a multiplicity of one. Assume that G possesses s ($s > 1$) connected components, each containing n_i vertices, where $n_i \geq 3$, for all $i = 1, 2, \dots, s$. By applying Lemma 4.4, we ascertain that $\text{rank}(\mathcal{L}(G)) = n - s$, indicating that the multiplicity of the eigenvalue 0 is s . □

4.2 Laplacian ISI-eigenvalues of some families of graphs

The Laplacian ISI-eigenvalues of specific families of graphs are something that will be discussed in this section. In this study, we extend the conclusions that were previously established for the Laplacian eigenvalues of classical Laplacians to the Laplacian ISI-eigenvalues of these graph families.

Theorem 4.9. *Let G be an n -vertex graph. If the graph G is r -regular, then $\eta_i = \frac{r}{2}\mu_i$ for all $1 \leq i \leq n$. Specifically, if G is isomorphic to K_n , then $\eta_1 = \eta_2 = \dots = \eta_{n-1} = \frac{n(n-1)}{2}$ and $\eta_n = 0$; if G is C_n , then $\eta_i = \mu_i$ for $i = 1, 2, \dots, n$.*

Proof. If G is r -regular, it follows that $\mathcal{L}(G) = \frac{r}{2}L(G)$, and thus, $\eta_i = \frac{r}{2}\mu_i$ for all i , satisfying the condition $1 \leq i \leq n$. This, together with the condition that if G is equal to K_n , then $\mu_1 = \mu_2 = \dots = \mu_{n-1} = n$ and $\mu_n = 0$, leads to $\eta_1 = \eta_2 = \dots = \eta_{n-1} = \frac{n(n-1)}{2}$ and $\eta_n = 0$. Furthermore, for $G = C_n$, it is established that $\mu_i = 2 - 2 \cos \frac{2\pi i}{n}$ for $i = 0, 1, \dots, n-1$, which would produce the desired outcome. □

Theorem 4.10. *Let G be a complete bipartite graph with cardinalities m and n for the two partitioned sets, respectively. Subsequently, the Laplacian ISI-spectrum of G for this graph is as follows:*

$$\text{Spec}(G) = \left\{ 0, \frac{n^n + m^3}{n+m}, \frac{n^3}{n+m}^{(m-1)}, \frac{m^3}{n+m}^{(n-1)} \right\}$$

Proof. Given that G is a complete bipartite graph with m and n cardinalities, the following is the definition of the ISI-Laplacian matrix of G :

$$\mathcal{L}(G) = \begin{pmatrix} \frac{n^3}{n+m} & 0 & \cdots & 0 & \frac{-nm}{n+m} & \frac{-nm}{n+m} & \cdots & \frac{-nm}{n+m} \\ 0 & \frac{n^3}{n+m} & \cdots & 0 & \frac{-nm}{n+m} & \frac{-nm}{n+m} & \cdots & \frac{-nm}{n+m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n^3}{n+m} & \frac{-nm}{n+m} & \frac{-nm}{n+m} & \cdots & \frac{-nm}{n+m} \\ \frac{-nm}{n+m} & \frac{-nm}{n+m} & \cdots & \frac{-nm}{n+m} & \frac{m^3}{n+m} & 0 & \cdots & 0 \\ \frac{-nm}{n+m} & \frac{-nm}{n+m} & \cdots & \frac{-nm}{n+m} & 0 & \frac{m^3}{n+m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-nm}{n+m} & \frac{-nm}{n+m} & \cdots & \frac{-nm}{n+m} & 0 & 0 & \cdots & \frac{m^3}{n+m} \end{pmatrix}.$$

To determine the eigenvalues of $\mathcal{L}(G)$, we set the characteristic polynomial of $\mathcal{L}(G)$ equal to zero, that is, $\det(\lambda I - \mathcal{L}(G)) = 0$. This gives:

$$\begin{vmatrix} \lambda - \frac{n^3}{n+m} & 0 & \cdots & 0 & \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} \\ 0 & \lambda - \frac{n^3}{n+m} & \cdots & 0 & \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - \frac{n^3}{n+m} & \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} \\ \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} & \lambda - \frac{m^3}{n+m} & 0 & \cdots & 0 \\ \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} & 0 & \lambda - \frac{m^3}{n+m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} & 0 & 0 & \cdots & \lambda - \frac{m^3}{n+m} \end{vmatrix} = 0.$$

Using the definition of the determinant for a block matrix, we articulate the previously stated equation:

$$\det(A) \cdot \det(D - CA^{-1}B) = 0, \quad (4.3)$$

$$\text{where } A = \begin{bmatrix} \lambda - \frac{n^3}{n+m} & 0 & \cdots & 0 \\ 0 & \lambda - \frac{n^3}{n+m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - \frac{n^3}{n+m} \end{bmatrix}_{m \times m}, \quad B = \begin{bmatrix} \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} \\ \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{nm}{n+m} & \frac{nm}{n+m} & \cdots & \frac{nm}{n+m} \end{bmatrix}_{m \times n},$$

$$C = B^t \text{ and } D = \begin{bmatrix} \lambda - \frac{m^3}{n+m} & 0 & \cdots & 0 \\ 0 & \lambda - \frac{m^3}{n+m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - \frac{m^3}{n+m} \end{bmatrix}_{n \times n}.$$

One can easily see that:

$$\det(A) = \left(\lambda - \frac{n^3}{n+m} \right)^m. \quad (4.4)$$

To determine the second determinant in the product $\det(A) \cdot \det(D - CA^{-1}B) = 0$, we first compute $CA^{-1}B$, that is,

$$\begin{aligned} CA^{-1}B &= \left(\frac{nm}{n+m} \right)^2 \left(\frac{1}{\lambda - \frac{n^3}{n+m}} \right) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times m} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{m \times m} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m \times n} \\ &= \left(\frac{nm}{n+m} \right)^2 \left(\frac{1}{\lambda - \frac{n^3}{n+m}} \right) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times m} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m \times n} \\ &= \left(\frac{nm}{n+m} \right)^2 \left(\frac{m}{\lambda - \frac{n^3}{n+m}} \right) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times n}. \end{aligned}$$

Next, we will find the matrix represented by $D - CA^{-1}B$, which is equal to:

$$D - CA^{-1}B = \begin{bmatrix} a-b & -b & -b & \cdots & -b \\ -b & a-b & -b & \cdots & -b \\ -b & -b & a-b & \cdots & -b \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b & -b & -b & \cdots & a-b \end{bmatrix}_{n \times n},$$

where $a = \lambda - \frac{m^3}{m+n}$ and $b = \left(\frac{nm}{n+m} \right)^2 \frac{m}{\lambda - \frac{n^3}{n+m}}$. Finally, the computation of the determinant of matrix $D - CA^{-1}B$ yields the following equation:

$$\det(D - CA^{-1}B) = \begin{vmatrix} a-b & -b & -b & \cdots & -b \\ -b & a-b & -b & \cdots & -b \\ -b & -b & a-b & \cdots & -b \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b & -b & -b & \cdots & a-b \end{vmatrix}_{n \times n}.$$

By summing all columns into the first column of the determinant and factoring out the common term from column one, we obtain:

$$\det(D - CA^{-1}B) = (a - nb) \begin{vmatrix} 1 & -b & -b & \cdots & -b \\ 1 & a-b & -b & \cdots & -b \\ 1 & -b & a-b & \cdots & -b \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & -b & -b & \cdots & a-b \end{vmatrix}_{n \times n}.$$

By subtracting the first row of the determinant from the subsequent rows and subsequently expanding the determinant, we derive the following:

$$\det(D - CA^{-1}B) = (a - nb)a^{n-1}. \quad (4.5)$$

By Combining equations (4.4) and (4.5), equation (4.3) can be rewritten as:

$$\det(\lambda I - \mathcal{L}(G)) = \left(\lambda - \frac{n^3}{n+m} \right)^m (a - nb)a^{n-1},$$

where $a = \lambda - \frac{m^3}{m+n}$ and $b = \left(\frac{nm}{n+m} \right)^2 \frac{m}{\lambda - \frac{n^3}{n+m}}$. By simplifying the aforementioned equation, we obtain:

$$\begin{aligned} \det(\lambda I - \mathcal{L}(G)) &= \left(\lambda - \frac{n^3}{n+m} \right)^m (a - nb)a^{n-1} \\ &= \left(\lambda - \frac{n^3}{n+m} \right)^m \left(\lambda - \frac{m^3}{m+n} - n \cdot \left(\frac{nm}{n+m} \right)^2 \frac{m}{\lambda - \frac{n^3}{n+m}} \right) \left(\lambda - \frac{m^3}{m+n} \right)^{n-1} \\ &= \lambda \left(\lambda - \frac{n^3}{n+m} \right)^{m-1} \left(\lambda - \left(\frac{m^3 + n^3}{m+n} \right) \right) \left(\lambda - \frac{m^3}{m+n} \right)^{n-1}. \end{aligned}$$

This completes the proof. $\square$

We define a graph $B_{m,n-m}$ that has a vertex set that is partitioned into two disjoint subsets, namely V_1 and V_2 , where V_1 is comprised of m vertices and V_2 is composed of $n-m$ vertices. While one category of edges connects every vertex in V_1 to every vertex in V_2 , the other category of edges interconnects all of the vertices that are contained inside V_2 . The edge set of the graph is comprised of two categories of edges.

Theorem 4.11. Take n and m as two positive integers, such that n is greater than 3, m is greater than 2, and $n - m \geq 0$. Then, the

$$\begin{aligned} \text{Spec}(B_{m,n-m}) &= \left\{ 0, \frac{m(n-1)^2 + (n-m)^3}{2n-m-1}, \frac{(n-m)^3}{2n-m-1} \right. \\ &\quad \left. (n-1) \left(\frac{(n-m-1)}{2} + \frac{m(n-1)}{2n-m-1} + \frac{1}{2} \right)^{(n-m-1)} \right\}. \end{aligned}$$

Proof. Let $B_{m,n-m}$ denote a graph as stated above. Then the ISI-Laplacian matrix of $B_{m,n-m}$ is presented as follows:

$$\mathcal{L}(B_{m,n-m}) = \begin{pmatrix} a & 0 & \cdots & 0 & -b & -b & \cdots & -b \\ 0 & a & \cdots & 0 & -b & -b & \cdots & -b \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \cdots & a & -b & -b & \cdots & -b \\ -b & -b & \cdots & -b & c & -d & \cdots & -d \\ -b & -b & \cdots & -b & -d & c & \cdots & -d \\ \vdots & \vdots & \ddots & \vdots & & & & \\ -b & -b & \cdots & -b & -d & -d & \cdots & c \end{pmatrix},$$

where $a = \frac{(n-m)^3}{2n-m-1}$, $b = \frac{(n-1)(n-m)}{2n-m-1}$, $c = \frac{(n-1)(n-m-1)}{2} + \frac{m(n-1)^2}{2n-m-1}$ and $d = \frac{n-1}{2}$. The characteristic polynomial of $\mathcal{L}(B_{m,n-m})$ is $\det(\lambda I - \mathcal{L}(B_{m,n-m})) = 0$. This gives:

$$\det(\lambda I - \mathcal{L}(B_{m,n-m})) = \begin{vmatrix} \lambda - a & 0 & \cdots & 0 & b & b & \cdots & b \\ 0 & \lambda - a & \cdots & 0 & b & b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - a & b & b & \cdots & b \\ b & b & \cdots & b & \lambda - c & d & \cdots & d \\ b & b & \cdots & b & d & \lambda - c & \cdots & d \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & b & d & d & \cdots & \lambda - c \end{vmatrix} = 0.$$

Using the definition of the determinant of a block matrix, we express the aforementioned determinant as follows:

$$\det(A) \cdot \det(D - CA^{-1}B) = 0 \quad (4.6)$$

$$\text{where } A = \begin{bmatrix} \lambda - a & 0 & \cdots & 0 \\ 0 & \lambda - a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - a \end{bmatrix}_{m \times m}, B = \begin{bmatrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & m \end{bmatrix}_{m \times n-m},$$

$$C = \begin{bmatrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & m \end{bmatrix}_{n-m \times m} \text{ and } D = \begin{bmatrix} \lambda - c & d & \cdots & d \\ d & \lambda - c & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & \lambda - c \end{bmatrix}_{n-m \times n-m}.$$

It is easy to see that

$$\det(A) = (\lambda - a)^m, \text{ where } a = \frac{(n-m)^3}{2n-m-1}. \quad (4.7)$$

To compute the second determinant in (4.6), we first evaluate the matrix $CA^{-1}B$.

$$\begin{aligned} CA^{-1}B &= \begin{bmatrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & m \end{bmatrix}_{n-m \times m} \cdot \begin{bmatrix} \lambda - a & 0 & \cdots & 0 \\ 0 & \lambda - a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - a \end{bmatrix}_{m \times m} \cdot \begin{bmatrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & m \end{bmatrix}_{m \times n-m} \\ &= b^2 \left(\frac{1}{\lambda - a} \right) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n-m \times m} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m \times n-m} \\ &= mb^2 \left(\frac{1}{\lambda - a} \right) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n-m \times n-m}. \end{aligned}$$

Subsequently, we will compute $D - CA^{-1}B$, which is equivalent to:

$$\begin{aligned}
 D - CA^{-1}B &= \begin{bmatrix} \lambda - c & d & \cdots & d \\ d & \lambda - c & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & \lambda - c \end{bmatrix}_{n-m \times n-m} - mb^2 \left(\frac{1}{\lambda - a} \right) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n-m \times n-m} \\
 &= \begin{bmatrix} \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) & d - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & d - mb^2 \left(\frac{1}{\lambda - a} \right) \\ d - mb^2 \left(\frac{1}{\lambda - a} \right) & \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & d - mb^2 \left(\frac{1}{\lambda - a} \right) \\ \vdots & \vdots & \ddots & \vdots \\ d - mb^2 \left(\frac{1}{\lambda - a} \right) & d - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) \end{bmatrix}_{n-m \times n-m}.
 \end{aligned}$$

The determinant of the matrix $D - CA^{-1}B$ is expressed in the following equation:

$$\det(D - CA^{-1}B) = \begin{vmatrix} \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) & d - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & d - mb^2 \left(\frac{1}{\lambda - a} \right) \\ d - mb^2 \left(\frac{1}{\lambda - a} \right) & \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & d - mb^2 \left(\frac{1}{\lambda - a} \right) \\ \vdots & \vdots & \ddots & \vdots \\ d - mb^2 \left(\frac{1}{\lambda - a} \right) & d - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) \end{vmatrix}_{n-m \times n-m}$$

First adding all columns into the first column and subsequently extracting the common term from the first column, we obtain following results:

$$\det(D - CA^{-1}B) = X \cdot \begin{vmatrix} 1 & d - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & d - mb^2 \left(\frac{1}{\lambda - a} \right) \\ 1 & \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & d - mb^2 \left(\frac{1}{\lambda - a} \right) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d - mb^2 \left(\frac{1}{\lambda - a} \right) & \cdots & \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) \end{vmatrix}_{n-m \times n-m},$$

where $X = \lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) + (n - m - 1) \left(d - mb^2 \left(\frac{1}{\lambda - a} \right) \right)$. Next, by subtracting the first row from all subsequent rows and then expanding the determinant, we derive the following equation:

$$\begin{aligned}
 &\det(D - CA^{-1}B) \\
 &= \left(\lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) + (n - m - 1) \left(d - mb^2 \left(\frac{1}{\lambda - a} \right) \right) \right) \cdot (\lambda - c - d)^{n-m-1} \quad (4.8)
 \end{aligned}$$

Combining equations (4.6) and (4.7), we derive the subsequent equation:

$$\det(\lambda I - B_{m,n-m}) = (\lambda - a)^m \cdot \left(\lambda - c - mb^2 \left(\frac{1}{\lambda - a} \right) + (n - m - 1) \left(d - mb^2 \left(\frac{1}{\lambda - a} \right) \right) \right) \cdot (\lambda - c - d)^{n-m-1}.$$

where $a = \frac{(n-m)^3}{2n-m-1}$, $b = \frac{(n-1)(n-m)}{2n-m-1}$, $c = \frac{(n-1)(n-m-1)}{2} + \frac{m(n-1)^2}{2n-m-1}$ and $d = \frac{n-1}{2}$. By substituting the values, we ultimately derive:

$$\begin{aligned}
 \det(\lambda I - B_{m,n-m}) &= \lambda \left(\lambda - \frac{(n-m)^3}{2n-m-1} \right)^{m-1} \left(\lambda - \frac{m(n-1)^2}{2n-m-1} - \frac{(n-m)^3}{2n-m-1} \right) \\
 &\quad \cdot \left(\lambda - (n-1) \left(\frac{n-m-1}{2} + \frac{m(n-1)^2}{2n-m-1} + \frac{1}{2} \right) \right)^{n-m-1}.
 \end{aligned}$$

This completes the proof. $\square$

A direct consequence of Theorem 4.10 and Theorem 4.11 is the subsequent corollary, which is provided herein.

Corollary 4.12. *For $n \geq 3$, the following holds true.*

$$\begin{aligned}\text{Spec}(K_{n,n}) &= \left\{ 0, n^2, \frac{n^{2^{(2n-2)}}}{2} \right\} \\ \text{Spec}(K_{n-1,1}) &= \left\{ 0, \frac{(n-1)^3}{n}, \frac{1^{(n-2)}}{n} \right\} \\ \text{Spec}(K_n - e) &= \left\{ 0, \frac{2(n-1)^2 + (n-2)^3}{2n-3}, \frac{(n-2)^3}{2n-3}, (n-1) \left(\frac{n-2}{2} + \frac{2(n-1)}{2n-3} \right)^{(n-3)} \right\}.\end{aligned}$$

The established theorem by Brouwer and Haemers [3] outlines a correlation between the diameter of a graph and the number of distinct eigenvalues that are present in the graph.

Theorem 4.13. [3] *Let G represent a connected graph with a diameter equal to D . Consequently, G possesses a minimum of $D + 1$ distinct adjacency eigenvalues (respectively, Laplacian eigenvalues), and no less than $D + 1$ distinct signless Laplacian eigenvalues.*

The proof presented in [3] confirms that the aforementioned claim is true for every symmetric matrix $M = (m_{ij})$ with non-negative entries corresponding to the vertices of a graph G , where $m_{ij} > 0$ if and only if an edge exists between vertices v_i and v_j . The next corollary is obtained immediately by analyzing the matrix $M = n^2I - \mathcal{L}(G)$.

Corollary 4.14. *If G is a graph with diameter D with k different Laplacian ISI-eigenvalues, then k must be greater than or equal to $D + 1$.*

Theorem 4.15. *Let G be an n -vertex connected graph, where $n \geq 4$. Consequently, the subsequent statements are valid.*

- (i) *No graph G can have three distinct Laplacian ISI eigenvalues if its diameter is at least 3.*
- (ii) *If G is a star graph or a complete bipartite graph with equal-sized partite sets, it has three Laplacian ISI-eigenvalues.*
- (iii) *A cycle G of length n has precisely three different Laplacian ISI-eigenvalues if and only if it is either C_4 or C_5 .*
- (iv) *If G is a regular graph that is not a complete graph K_n , then G has three different Laplacian ISI-eigenvalues if and only if G is a strongly regular graph.*

Proof. (i). Let G be a graph such that its diameter is no less than 3, then according to Corollary 4.14, G possesses more than three distinct Laplacian ISI-eigenvalues.

(ii). If G is $K_{1,n-1}$, then by Corollary 4.12, we find that $\text{Spec}(K_{n-1,1}) = \left\{ 0, \frac{(n-1)^3}{n}, \frac{1^{(n-2)}}{n} \right\}$. It is clear that G has exactly three unique Laplacian ISI-eigenvalues. If G is $K_{p,p}$ with $n = 2p$,

then, as per Corollary 4.12, the Laplacian ISI-spectrum of G is expressed as $\text{Spec}(K_{n,n}) = \left\{0, n^2, \frac{n^2}{2}^{(2n-2)}\right\}$, and this finding is relevant in this context.

Conversely, let G be a bipartite graph with a diameter of at most two that has three distinct Laplacian ISI-eigenvalues. We claim that G is either $K_{p,p}$ or $K_{n-1,1}$. It is clear that K_n is the only connected graph with a diameter of 1, and as per Theorem 4.7, this graph has two unique Laplacian ISI-eigenvalues. Therefore, G cannot have a diameter of 1. Therefore, the diameter of G must equal 2. Let G be a bipartite graph that has a diameter of 2. Let u and v be two vertices of the graph G that are not next to each other. If u has a neighbor that is not adjacent to v , then this neighbor, along with u and v , creates a path P_4 . This means that the diameter of G is more than 2, which is not possible. So, any two vertices that aren't adjacent to one other must have a common neighbor. Consequently, G constitutes a complete bipartite graph. In the context of a complete bipartite graph, if G is either $K_{p,p}$ or $K_{n-1,1}$, no demonstration is required. Conversely, if G is $K_{m,n-m}$ with $m \neq 1$ and $m \neq \frac{n}{2}$, Theorem 4.10 asserts that this graph has four unique Laplacian ISI-eigenvalues. This completes the proof in this case.

(iv). Let G be a k -regular graph. Then in this case, we have $d_i = k$ for every i . The Laplacian ISI-matrix is expressed as $\mathcal{L}(G) = \frac{r}{2}L(G)$. This indicates that $\mathcal{L}(G)$ has three different eigenvalues if and only if $L(G)$ has three distinct eigenvalues. Based on the premise that regular graphs with three distinct Laplacian eigenvalues are precisely strongly regular graphs, the conclusion is drawn in this case. This completes the proof. $\square$

5 Experimental Setup

The experimental evaluation is conducted using synthetically generated graph structures to model IoT-like network environments under both normal and anomalous conditions. Three representative graph models—Erdős–Rényi, Watts–Strogatz, and Barabási–Albert—are utilized to capture random, small-world, and scale-free connectivity patterns respectively. These graphs serve as baseline network topologies for simulating realistic communication structures in IoT and cyber-physical systems. Normal network conditions are represented by unperturbed graphs with stable connectivity patterns and consistent degree distributions. To simulate anomalous behavior, structural perturbations are introduced by modifying the edge set of the original graphs. This includes the addition of spurious edges between randomly selected node pairs, representing malicious link injections, unauthorized communications, or compromised device interactions. In some cases, node-level connectivity is also altered to reflect abnormal behavioral changes in network entities. These modifications intentionally disrupt the inherent structural properties of the original graph, leading to changes in connectivity distribution and local neighborhood relationships.

6 Results and Discussion

Results show that spectral deviations effectively indicate anomalies. The ISI-based approach provides improved sensitivity compared to classical methods. The experimental

results demonstrate the effectiveness of Laplacian eigenvalue spectra for distinguishing between normal and anomalous network structures in an IoT-like graph setting. A baseline Erdős–Rényi graph is used to represent normal network behavior, while anomalies are introduced by adding a few extra edges to simulate structural perturbations as shown in figure 3. The eigenvalues of the graph Laplacian are computed for both normal and anomalous graphs, and their distributions are compared. It is observed that the normal graph exhibits a smooth and stable eigenvalue pattern, reflecting uniform random connectivity. In contrast, the anomalous graph shows noticeable deviations in its eigenvalue spectrum due to changes in node degrees and connectivity structure. These deviations confirm that even minor struc-

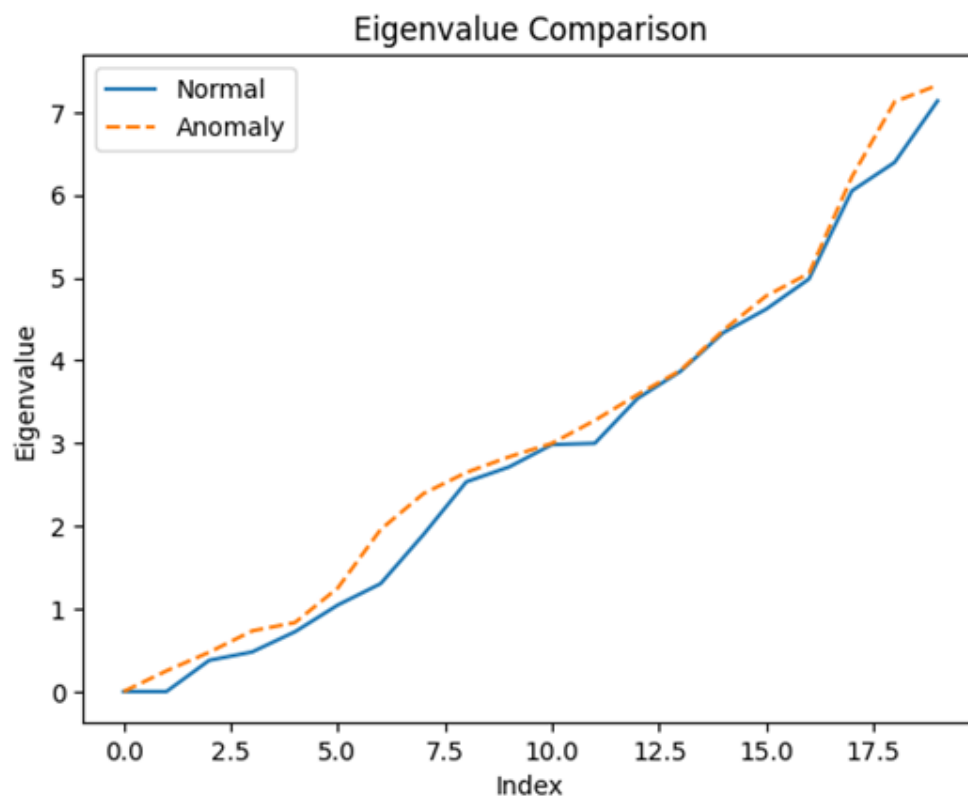


Figure 3: Eigenvalue Comparison

tural modifications in a graph lead to measurable changes in Laplacian eigenvalues. The spectral differences are further quantified using the Euclidean distance between eigenvalue vectors, producing an anomaly score. A higher anomaly score indicates greater structural divergence between the two graphs, validating the sensitivity of the method. The plotted results visually confirm this behavior, where the anomalous curve diverges from the normal curve across multiple eigenvalue indices. Overall, the findings demonstrate that Laplacian eigenvalue-based spectral analysis provides a lightweight and effective mechanism for detecting structural anomalies in complex networks.

Table 1 quantifies anomaly detection performance using spectral distance between normal and anomalous eigenvalue distributions. The Euclidean norm of eigenvalue differences is used as the anomaly score, where higher values indicate greater structural deviation. Results

Network Model	Normal Eigen Stability	Anomalous Eigen Shift	Spectral Distance Score	Detection Strength
Erdos–Renyi	High	Low	Low	Weak–Moderate
Watts–Strogatz	High	Medium	Medium	Strong
Barabási–Albert	Medium	High	High	Very Strong

Table 1: Comparison of Network Models Based on Spectral Properties

demonstrate that small-world and scale-free networks are more sensitive to structural perturbations compared to random graphs. This confirms that Laplacian eigenvalues provide a reliable global signature for detecting anomalies in complex IoT network structures.

The experimental study employs three fundamental graph models—Erdős–Rényi, Watts–Strogatz, and Barabási–Albert—to evaluate the effectiveness of Laplacian eigenvalue-based anomaly detection across different network topologies. The Erdős–Rényi graph represents a random network where edges are formed independently with a fixed probability, making it suitable for modeling baseline or uncontrolled IoT environments with uniform but unstructured connectivity as shown in figure 4. In contrast, the Watts–Strogatz small-world graph introduces high clustering with short average path lengths, closely resembling real IoT and sensor networks where local communication is dominant but occasional long-range connections exist. The Barabási–Albert scale-free graph captures power-law degree distribution, where a few hub nodes have significantly higher connectivity, reflecting real-world Internet, cloud, and cyber-physical systems. These hubs are critical points in security analysis, as they are highly vulnerable to targeted attacks or failures.

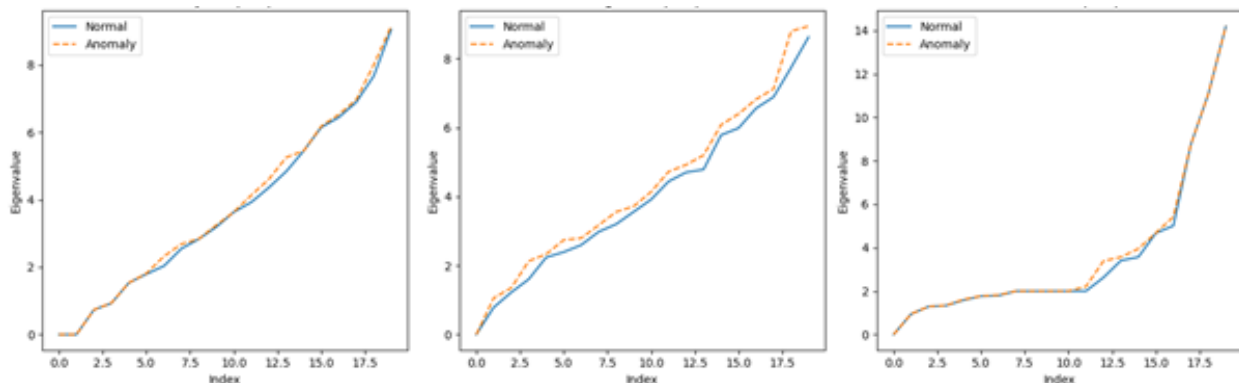


Figure 4: Laplacian Eigenvalue-Based Anomaly Detection Across Random, Small-World, and Scale-Free Network Models

When anomaly edges are introduced into each graph, they simulate structural perturbations such as malicious link injection, unauthorized communication, or compromised device interactions. In the Erdős–Rényi model, anomalies slightly disturb the uniform randomness, leading to mild but measurable spectral shifts. However, in the Watts–Strogatz model, added edges disrupt the balance between clustering and regularity, producing more pronounced changes in the Laplacian spectrum due to its sensitivity to local neighborhood structure. In the Barabási–Albert network, anomalies affecting hub-related nodes cause sig-

nificant spectral distortion because eigenvalues in scale-free networks are highly influenced by high-degree nodes. Across all three models, the Laplacian eigenvalues serve as global structural fingerprints, and deviations in their distribution directly reflect changes in connectivity patterns. The computed anomaly scores confirm that scale-free and small-world networks exhibit higher sensitivity to structural perturbations compared to purely random graphs. Overall, the comparative analysis demonstrates that spectral graph methods are robust across heterogeneous IoT network models and effectively capture both local and global structural anomalies.

7 Conclusion

This paper presents a spectral graph-theoretic framework for anomaly detection in Internet of Things (IoT) networks using Laplacian Inverse Sum Indegree (ISI) eigenvalues. The study models IoT environments as complex graphs, where nodes represent devices and edges represent communication links. To capture structural and degree-sensitive relationships more effectively than classical methods, the Laplacian ISI matrix is constructed and its eigenvalues are extracted as global spectral features. These eigenvalues provide a compact representation of network topology and are highly sensitive to structural perturbations. The proposed approach analyzes deviations in the eigenvalue spectrum between normal and anomalous network states. Anomalies such as malicious link injection, compromised device communication, and structural disruptions lead to measurable changes in the spectral distribution. The framework is evaluated on multiple graph models, including Erdős–Rényi, Watts–Strogatz, and Barabási–Albert networks, representing random, small-world, and scale-free IoT structures. Results demonstrate that scale-free and small-world networks exhibit higher sensitivity to anomalies compared to random graphs. To quantify anomaly detection, spectral distance measures are used to compute deviation between normal and attack scenarios. The findings confirm that Laplacian ISI eigenvalues effectively capture both local and global structural changes in IoT networks. Overall, the study bridges theoretical spectral graph concepts with practical cybersecurity applications, offering a lightweight, interpretable, and scalable method for anomaly detection in complex networked systems.

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